Mathematical modeling of dendrite generators

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Abstract. A macroscopic model for the transport and the phase transitions of water in dendrite snow generators is derived. New aspects compared to results from the literature are a new approximate model for evaporation/condensation and a model for snow flakes based on a monodisperse size distribution.

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1 Introduction

We assume the air mass density ϱ, the pressure p, the temperature T, and the velocity v as given functions of position x and time t. Inside the container water occurs as vapor, liquid droplets, and snow flakes. In particular, we treat ice particles as (small) snow flakes immediately after entering the dendrite generator, therefore the ice particles are only described as inflow of snow flakes with radius zero.

We shall derive a model for the transport of water in its different forms, for phase transitions, and for the corresponding energy gains/losses. The phase transition processes considered here are collected in the following table. Note that we are interested in cold environments, where transitions from the solid to the liquid phase can be neglected.

	vapor	droplets	snow
vapor	x	evaporation	evaporation
droplets	condensation	x	x
snow	deposition	collection	x

2 Balance equations for particles with size distribution (snow flakes and droplets)

We denote by $f_j(x,R,t)$, $j = s,d$, the distribution functions for snow and droplets with respect to position and radius R. Whereever shape plays a role, snow flakes and droplets will be assumed as spherical. Assuming that particles move and change their size, but are not created or destroyed, the distribution function solves the transport equation

$$\partial_t f_j + \nabla \cdot (v_j f_j) + \partial_R(\dot{R}_j f_j) = 0\,, \tag{1}$$

where the convective velocity is given by

$$v_j(x,R,t) = v(x,t) - V_j(R)e_z\,,$$

with the size dependent fall speed of snow flakes V_s given below, $V_d = 0$, and $e_z = (0,0,1)$. The growth term $\dot{R}_j$ results from phase transitions and will be derived below.

From the distribution function, the number and mass densities can be computed:

$$n_j(x,t) = \int_0^\infty f_j(x,R,t)dR\,, \qquad \varrho_j(x,t) = \int_0^\infty m_j(R) f_j(x,R,t)dR\,,$$

with the mass $m_j(R) = \varrho_{j,0}4\pi R^3/3$ of a particle with radius R, where $\varrho_{s,0}$ and $\varrho_{d,0}$ denote the densities of ice and liquid water, respectively. Equations for the position densities are derived by integration of (1):

$$\partial_t n_j + \nabla \cdot \left(\int_0^\infty v_j f_j\, dR \right) = 0\,, \tag{2}$$

$$\partial_t \varrho_j + \nabla \cdot \left(\int_0^\infty m_j v_j f_j\, dR \right) = \int_0^\infty m_j' \dot{R}_j f_j\, dR\,. \tag{3}$$

Finally, we also derive the equation

$$\partial_t(n_j R_j) + \nabla \cdot \left(\int_0^\infty R v_j f_j \, dR \right) = \int_0^\infty \dot{R}_j f_j \, dR \,, \tag{4}$$

for the average radius defined by

$$R_j(x,t) = \frac{1}{n_j(x,t)} \int_0^\infty R f_j(x,R,t) dR \,.$$

3 Microphysics of the phase changes

Evaporation and condensation/deposition

In this section the phase transitions between vapor and liquid water on the one hand, and vapor and ice on the other hand will be modelled.

We consider a spherical particle (a droplet or snow flake) of radius R placed in an environment filled with vapor. We assume that this microscopic problem is spherically symmetric and consider the microscopic radius variable r ranging from the surface $r = R$ of the particle to the far field $r = \infty$.

The vapor density ϱ_v solves a quasistationary diffusion problem, i.e. the Laplace equation in spherical coordinates, hence it is given by

$$\varrho_v(r) = \varrho_v(\infty) - \frac{R}{r}(\varrho_v(\infty) - \varrho_v(R)) \,.$$

Then the change of mass of the particle is given by

$$\dot{m}_{j,v} = 4\pi R^2 D_v \varrho_v'(R) = 4\pi R D_v (\varrho_v(\infty) - \varrho_v(R)) \,, \tag{5}$$

where D_v is the vapor diffusivity. If $\varrho_v(\infty) > \varrho_v(R)$ vapor diffuses inward and condenses on the surface, if $\varrho_v(\infty) < \varrho_v(R)$ evaporation occurs and diffusion away from the surface. The change of energy due to the phase transition can be derived in a similar manner and is given by

$$L_j \dot{m}_{j,v} = 4\pi\kappa R (T(R) - T(\infty)), \tag{6}$$

where κ is the thermal conductivity of air (almost independent of humidity, however somewhat dependent on temperature, which is ignored here, and the value for 0^oC is used), and L_s and L_d are the latent heats of the evaporation of ice and liquid water, respectively. If these are assumed constant, the Clausius-Capeyron equation for the saturation pressure leads to

$$p_{s,j}(T) = p_{s,j,0} \exp\left(\frac{L_j}{R_v} \left(\frac{1}{T_0} - \frac{1}{T} \right) \right) \,,$$

where $p_{s,j,0}$ is the measured saturation pressure at the reference temperature T_0, typically chosen as corresponding to zero degrees Celsius, and R_v is the vapor gas constant. The ideal gas law for vapor then provides the saturation vapor densities

$$\varrho_{v,s,j}(T) = \frac{p_{s,j}(T)}{R_v T} \,.$$

With the assumption that the vapor density takes its saturation value at the surface of the particle, elimination of $\dot{m}_{j,v}$ from (5) and (6) leads to an equation for $T(R)$:

$$L_j D_v \left[\varrho_v(\infty) - \varrho_{v,s,j}(T(R))\right] = \kappa \left[T(R) - T(\infty)\right] . \tag{7}$$

Linearization around $T(\infty)$ gives the approximation

$$T(R) - T(\infty) = \frac{L_j D_v[\varrho_v(\infty) - \varrho_{v,s,j}(T(\infty))]}{\kappa + L_j D_v \varrho'_{v,s_j}(T(\infty))} .$$

Thus, the change of mass of a particle with radius R can be computed in terms of the far-field values of the vapor density and the temperature:

$$\dot{m}_{j,v}(R) = \frac{4\pi\kappa R D_v}{\kappa + L_j D_v \varrho'_{v,s,j}(T(\infty))} \left[\varrho_v(\infty) - \varrho_{v,s,j}(T(\infty))\right] . \tag{8}$$

Collection of droplets by snow

We assume that this process (also called *riming*) occurs with 100% efficiency, i.e. whenever a droplet hits a snow flake, it immediately freezes and becomes part of the latter. We also assume that droplets are small compared to snow flakes. This means that for an individual snow flake of radius R, the droplets can be assumed as continually distributed. The growth of the snow flake mass by riming is therefore given as the product of the droplet density, the cross section area of the snow flake, and the modulus of the relative velocity $|v_s - v_d| = V_s$:

$$\dot{m}_{s,d}(R) = \varrho_d R^2 \pi V_s .$$

4 The macroscopic model

Snow flakes

If the air flow in the dendrite generator is laminar, snow flakes in a particular volume element at a particular time share their history. Therefore, a monodisperse distribution is assumed as a closure assumption for the macroscopic model:

$$f_s(x, R, t) = n_s(x,t)\delta(R - R_s(x,t)) ,$$

where δ denotes the Delta distribution. The radius growth term can be computed from the mass growth terms for evaporation/deposition and riming:

$$\dot{R}_s(R) = \frac{\dot{m}_{s,v}(R) + \dot{m}_{s,d}(R)}{4\pi R^2 \varrho_{s,0}} = \frac{\kappa D_v[\varrho_v - \varrho_{v,s,s}(T)]}{R\varrho_{s,0}[\kappa + L_s D_v \varrho'_{v,s,s}(T)]} + \frac{\varrho_d V_s}{4\varrho_{s,0}}$$

Using these in (2) and an appropriate combination of (2) and (4) gives

$$\partial_t n_s + \nabla \cdot [(v - V_s(R_s)e_z)n_s] = 0 ,$$

$$\partial_t R_s + (v - V_s(R_s)e_z) \cdot \nabla R_s = \frac{\kappa D_v[\varrho_v - \varrho_{v,s,s}(T)]}{R_s \varrho_{s,0}[\kappa + L_s D_v \varrho'_{v,s,s}(T)]} + \frac{\varrho_d V_s}{4\varrho_{s,0}} .$$

Droplets

For the droplets we make the popular assumption of a gamma size distribution:

$$f_d(x, R, t) = \frac{\varrho_d(x,t)R}{\pi \varrho_{d,0} R_{av}^5} e^{-R/R_{av}} \,,$$

with a prescribed average radius R_{av}. The contributions to $\dot{R}_d$ from evaporation/condensation and riming are obtained analogously to the above, whence (3) with $j = d$ becomes

$$\partial_t \varrho_d + \nabla \cdot (v \varrho_d) = \frac{\varrho_d \kappa D_v (\varrho_v - \varrho_{v,s,d}(T))}{\varrho_{d,0} R_{av}^2 (\kappa + L_d D_v \varrho'_{v,s,d}(T))} - n_s \varrho_d R_s^2 \pi V_s \,.$$

Vapor

Vapor is also convected with the air flow. All the evaporation and condensation/deposition effects have to be taken into account:

$$\partial_t \varrho_v + \nabla \cdot (v \varrho_v) = \frac{\varrho_d \kappa D_v (\varrho_{v,s,d}(T) - \varrho_v)}{\varrho_{d,0} R_{av}^2 \kappa + L_d D_v \varrho'_{v,s,d}(T)} + \frac{4\pi\kappa R_s n_s D_v (\varrho_{v,s,s}(T) - \varrho_v)}{\kappa + L_s D_v \varrho'_{v,s,s}(T)}$$

5 Model summary, energy balance, and parameter values

We collect the macroscopic model equations with a change of variables. Since in computations snow flakes start with radius $R_s = 0$, the right hand side of the R_s-equation becomes singular. As a remedy, we replace the radius of snow flakes by their surface area $A_s = 4\pi R_s^2$. The equations then become

$$\begin{aligned}
&\partial_t n_s + \nabla \cdot [(v - V_s e_z) n_s] = 0 \,, \\
&\partial_t A_s + (v - V_s e_z) \cdot \nabla A_s = \frac{4}{\varrho_{s,0}} \sqrt{\frac{\pi}{A_s}} \left(G_{s,v} + G_{s,d}\right) , \\
&\partial_t \varrho_d + \nabla \cdot (v \varrho_d) = G_{d,v} - G_{s,d} \,, \\
&\partial_t \varrho_v + \nabla \cdot (v \varrho_v) = -G_{d,v} - G_{s,v} \,,
\end{aligned}$$

with the rates for condensation/evaporation,

$$G_{d,v}(\varrho_v, \varrho_d, T) = \frac{\varrho_d \kappa D_v (\varrho_v - \varrho_{v,s,d}(T))}{\varrho_{d,0} R_{av}^2 (\kappa + L_d D_v \varrho'_{v,s,d}(T))} \,,$$

deposition/evaporation,

$$G_{s,v}(\varrho_v, n_s, A_s, T) = \frac{2\kappa \sqrt{\pi A_s}\, n_s D_v (\varrho_v - \varrho_{v,s,s}(T))}{\kappa + L_s D_v \varrho'_{v,s,s}(T)} \,,$$

and collection of snow from droplets,

$$G_{s,d}(\varrho_d, n_s, A_s) = \frac{n_s \varrho_d A_s V_s}{4} \,.$$

The fall speed of snow flakes strongly depends on their shape. Popular models are written in terms of a diameter $D_s = 2R_s = \sqrt{A_s/\pi}$ and have the form (see [2], p. 164)

$$V_s = V_{s,0} \left(\frac{D_s}{D_0}\right)^b \,.$$

We also recall the saturation vapor densities

$$\varrho_{v,s,j}(T) = \frac{p_{s,j,0}}{R_v T} \exp\left(\frac{L_j}{R_v}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right), \qquad j = s, d\,,$$

and their derivatives

$$\varrho'_{v,s,j}(T) = \left(\frac{L_j}{R_v} - T\right)\frac{\varrho_{v,s,j}(T)}{T^2}\,.$$

Since L_j/R_v is of the order of 5000K, the derivatives will remain positive in the temperature range considered here.

The energy produced/consumed by the phase changes per volume and time is given by

$$L_d G_{d,v} + L_s G_{s,v} + L_f G_{s,d}\,,$$

with the latent heat of freezing L_f. The thermal energy $E_{th} = (c\varrho + c_v\varrho_v)T$ is the sum of contributions from air and vapor, with the specific heats c of dry air and c_v of vapor. Neglecting the effects of viscosity, the heat balance equation reads

$$\partial_t E_{th} + \nabla \cdot (vE_{th} - \kappa \nabla T) = L_d G_{d,v} + L_s G_{s,v} + L_f G_{s,d}\,.$$

Boundary conditions

At the inflow boundary, the value of n_s has to be prescribed, representing the number density of ice particles. They are assumed to be very small. Therefore, $A_s = 0$ is taken as boundary value for the snow flake surface area. The boundary values of ϱ_d and ϱ_v describe the air-water mixture at the inflow.

References

[1] R.A. Houze jr., *Cloud Dynamics*, Academic Press, 1993.

[2] R.R. Rogers, M.K. Yau, *A Short Course in Cloud Physics*, Pergamon Press, 3rd ed., 1989.

Symbol	Description	value	unit
κ	thermal conductivity of air	2400	cm g s^{-3} K^{-1}
L_s	latent heat of deposition	2.834×10^{10}	cm^2 s^{-2}
L_d	latent heat of condensation	2.501×10^{10}	cm^2 s^{-2}
L_f	latent heat of freezing	0.334×10^{10}	cm^2 s^{-2}
D_v	diffusivity of vapor	2×10^{-5}	m^2 s^{-1}
R_v	gas constant of vapor	4.61×10^{6}	cm^2 s^{-2} K^{-1}
$\varrho_{s,0}$	density of ice	0.917	g cm^{-3}
$\varrho_{d,0}$	density of liquid water	1	g cm^{-3}
R_{av}	average droplet radius	2×10^{-3}	cm
T_0	reference temperature	273.16	K
$p_{s,s,0}$	reference vapor saturation pressure over ice	6111	g cm^{-1} s^{-2}
$p_{s,d,0}$	reference vapor saturation pressure over liquid water	6112	g cm^{-1} s^{-2}
$V_{s,0}$	reference fallspeed of snow	200	cm s^{-1}
D_0	reference snow diameter	1	cm
b	exponent in fallspeed formula	0.3	1
c	specific heat of dry air	1,005	cm^2 s^{-2} K^{-1}
c_v	specific heat of vapor	1,82	cm^2 s^{-2} K^{-1}

Table 1: Parameters